work, "New elements of geometry, with complete theory of parallels." Only from the 'New Elements' can any adequate idea be obtained of the height, the breadth, the depth of Lobachévski's achievement in the new universe of his own creation.

Of equal importance is the fact that Engel's book gives to the world at last à complete, available text-book of non-Euclidean geometry. There is no other to compare with it.

For the history of non-Euclidean geometry we have the admirable chapter X of Loria's pregnant work "Il passato ed il presente delle principali teorie geometriche." This chapter cites about 80 authors, mostly of writings devoted to non-Euclidean geometry.

In my own "Bibliography of hyper-space and non-Euclidean Geometry," in the *American Journal of Mathematics* (1878), I gave 81 authors and 174 titles. This when reprinted in the Collected Works of Lobachévski (Kazan, 1886) gives 124 authors and 272 titles.

Robert Bonola has just given in the Bollettino di Bibliografia e Storia della Scienze Matematiche (1899) an exceedingly rich and valuable "Bibliografia sui Fondamenti della Geometria in relazione alla Geometria Non-Euclidea," in which he gives 353 titles.

This extraordinary output of human thought has henceforth to be reckoned with. Hereafter no one may neglect it who attempts to treat of fundamentals in geometry or philosophy.

Austin, Texas, August 14, 1899.

NOTE ON THE LOXODROMICS OF THE SPHERE.

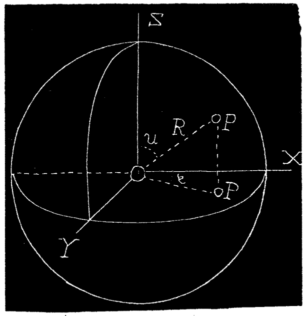
By HERMANN EMCH, of the University of Berne, Switzerland.

1. There is hardly a geometrical problem which from a didactic and practical point of view, and as an application of elementary calculus, is more valuable than that of the loxodromics of the sphere. In Nos. 6-7 of THE AMERICAN MATHEMATICAL MONTHLY (1899), Prof. G. B. M. Zerr proposed the problem to find the length of a N. W.-loxodromic of the earth-surface between the equator and certain parallels and meridians. This note has been prepared in view of this proposition and its value.

2. Let ε be the longitude and u the co-latitude of a point P on the surface of the earth supposed to be spherical and of radius R (Fig. 1).

The Cartesian coördinates x, y, z of P with regard to the planes of the 90°-meridian, the zero-meridian and the equator are, since $OP'=r=R\sin u$,

$$\left.\begin{array}{l} x=R.\sin u.\cos \varepsilon \\ y=R.\sin u.\sin \varepsilon \\ z=R.\cos u \end{array}\right\} \ldots\ldots(1).$$



The square of the linear element of the sphere is

$$ds^2=dx^2+dy^2+dz^2,$$

or since

$$dx=R^2(-\sin u.\sin \varepsilon.d\varepsilon+\cos u.\cos \varepsilon.du),$$
$$dy=R^2(\sin u.\cos \varepsilon.d\varepsilon+\cos u.\sin \varepsilon.du),$$
$$dz=R^2(-\sin u.du),$$

after some calculations;

$$ds^2=R^2.\sin^2 u\left(d\varepsilon^2+\frac{du^2}{\sin^2 u}\right)\ldots\ldots(2).$$

Putting $\int \frac{du}{\sin u}=kx,\ \varepsilon=ky,$ or $kx=\log\tan\frac{1}{2}u\ldots\ldots(3),\ ky=\varepsilon\ldots\ldots(4),$

where k designates any constant, we have

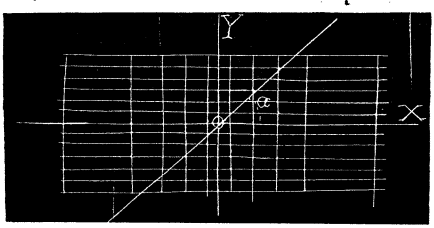
$$ds^2=R^2.k^2.\sin^2 u(dx^2+dy^2).$$

From (2), $\sin^2 u=\frac{4e^{2kx}}{(1+e^{2kx})^2}$, hence $ds^2=R^2.\frac{4k^2.e^{2kx}}{(1+e^{2kx})^2}(dx^2+dy^2)\ldots\ldots(5).$

3. Considering x and y as Cartesian coördinates of a plane (Fig. 2), it is seen that the surface of the sphere is mapped upon the plane by means of the formulas (2) and (3). The linear element of the XY-plane being $ds'=\sqrt{dx^2+dy^2}$, it follows that the ratio ds/ds' of two corresponding elements on the sphere, and the plane,

$$\frac{ds}{ds'}=\frac{2k.e^{kx}}{1+e^{2kx}}\ldots\ldots(6),$$

for infinitely small variations of x is constant, so that the surface of the sphere is conformally mapped upon the plane.



As it is well known this representation is called Mercator-projection. For constant values for u and ε the system of parallels, and meridians, arises.

Parallels if u is constant.

Meridians if ε is constant.

In the Mercator-projection to the parallels and meridians of the globe correspond two systems of parallel lines perperdicular to each other. According to the conformal projection any two lines on the sphere including a certain angle are

projected into two lines including the same angle. Consequently a loxodromic line on the sphere, *i. e.*, a curve which includes constant angles with the meridians is transformed into a straight line, and conversely to a straight line of the Mercator-projection corresponds a loxodromic line on the sphere.

In order to obtain the equation of the loxodromic line assume the two perpendicular lines as Cartesian axes, which correspond to the equator and the new meridian, respectively.

For the equator, $u=\frac{1}{2}\pi$, so that $kx=\log\tan\frac{1}{2}u=\log\tan\frac{1}{4}\pi=0$, or $x=0$, which represents the Y-axis. For the zero-meridian, $\varepsilon=0$, hence $y=0$, so that the X-axis corresponds to the zero-meridian. To a straight line $y=mx$ corresponds the loxodromic line $\varepsilon=m.\log\tan\frac{1}{2}u$, or

$$\tan\tfrac{1}{2}u=e^{\varepsilon/m}\ldots\ldots(7).$$

This loxodromic passes through the point of intersection of the equator and the zero-meridian, and includes an angle α with the meridian whose trigonometric tangent is m. The equation shows that the curve is winding an infinite number of times around the poles.

4. The projection of the loxodromic line upon the plane of the equator is a symmetrical double spiral, whose equation is obtained by substituting the value of $\sin u$ in the expression $r=R.\sin u$.

Now $\tan\frac{1}{2}u=e^{\varepsilon/m}$, from which

$$\sin u=\frac{2e^{\varepsilon/m}}{1+e^{2\varepsilon/m}},\quad\text{hence } r=\frac{2R.e^{\varepsilon/m}}{1+e^{2\varepsilon/m}},\quad\text{or } r=\frac{2R}{e^{\varepsilon/m}+^{-(\varepsilon/m)}}\ldots\ldots(8).$$

From this equation it is seen that the projected curve is symmetrical with regard to the zero-meridian.

5. To find the length of the loxodromic line between the equator and a parallel, for which $u=b$, we have

$$y=x\tan\alpha,\text{ or } y=xm,$$

$$dy=m.dx,$$

$$dy=m.\frac{du}{\sin u},$$

$$ds^2=R^2.\sin^2 u\left(\frac{du^2}{\sin^2 u}+m^2\frac{du^2}{\sin^2 u}\right),$$

$$ds=R.\sqrt{1+m^2}.du,\text{ and finally,}$$

$$S=R\int_b^{\frac{1}{2}\pi}\sqrt{1+m^2}.du=R\frac{\frac{1}{2}\pi-b}{\cos\alpha}\ldots\ldots(9).$$

This result may be stated in the theorem :

The length of a loxodromic whose inclination with the meridians is α, between two parallels with the latitudes a and b, is equal to the hypothenuse of a right triangle with the difference $(b-a)$ of the latitudes as one side and α as the adjacent angle.

6. If the length of the loxodromic between the equator and the meridian has to be found the element ds must be expressed by the variable ε. The equation of the Mercator-projection of the loxodromic being $\gamma = mx$, or on the sphere $\varepsilon = m \log \tan \frac{1}{2}u$, it follows that

$$d\varepsilon = m.\frac{du}{\sin u}, \text{ or } \frac{du}{\sin u} = \frac{d\varepsilon}{m}, \text{ thus } ds^2 = R^2 \sin^2 u.\left(\frac{d\varepsilon^2}{m^2} + d\varepsilon^2\right), \text{ or}$$

$$ds^2 = R^2 \sin^2 u\left(\frac{1+m^2}{m^2}\right) d\varepsilon^2.$$

Now $\sin^2 u = \frac{e^{2\varepsilon/m}}{(1+e^{2\varepsilon/m})^2}$, consequently $ds = R.\sqrt{\frac{1+m^2}{m^2}}.\frac{e^{\varepsilon/m}}{1+e^{2\varepsilon/m}}.d\varepsilon$,

$$\text{and } S = R.\int_0^{\varepsilon} \sqrt{\frac{1+m^2}{m^2}}.\frac{e^{\varepsilon/m}}{1+e^{2\varepsilon/m}}.d\varepsilon \ldots\ldots (10).$$

The value of this integral is:

$$S = R.\sqrt{1+m^2}.(\text{arc} \tan e^{\varepsilon/m} - \tfrac{1}{4}\pi) \ldots\ldots (11).$$

The length of the loxodromic line between $2(k-1)\pi$ and $2k\pi$, or of the kth winding, is

$$S_k = R.\sqrt{1+m^2}(\text{arc} \tan e^{2k\pi/m} - \text{arc} \tan e^{2(k-1)\pi/m}).$$

This value decreases as k increases and vanishes for $k = \infty$. For $k = 1, 2, 3, \ldots\ldots \infty$, $S = S_1 + S_2 + S_3 + \ldots\ldots$ ad infinitum, and appears as a convergent series, as it may easily be verified.

7. Numerical examples concerning a loxodromic, having an inclination with the meridian of 45°.

Radius of earth = 3956 miles.

A. Length of loxodromics from equator to parallels 30, 45, 60, 90, in miles.

Latitude.	Length of Loxodromic.
0	0
30	2929
45	4394
60	5859
90	8788

B. Length of loxodromics from equator and zero-meridian to meridians 90, 180, 270, 360 in miles.

Longitude.	Length of Loxodromic.
0	0
90	3247
180	4152
270	4344
360	4384

The entire length of the loxodromic is $S=\sqrt{(2)}\frac{1}{2}\pi.R=8788$ miles, which is obtained by putting in (9) $b=0$, or in (11) $\varepsilon=\infty$. This result coincides with the one obtained in the first table, where the length of the loxodromic for the latitude of 90° is also 8788 miles.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

114. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy, Irving College, Mechanicsburg, Pa.

Does it pay a \$4-carpenter using a dozen four-penny nails per minute, to pick up a dropped nail? At this rate, should twenty-penny nails be picked up?

Solution by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

The price of four-penny nails, at the present time, is 5 cents per pound. Assume that there are 200 nails to the pound, and that it takes the carpenter 10 seconds to pick up a nail.

The value of a nail is $\frac{5}{200}$ of a cent, or $\frac{1}{40}$ of a cent.

If we assume that the carpenter gets \$4.00 per day, and works 10 hours in a day, his wages is 40 cents per hour, or $\frac{1}{90}$ of a cent per second.

Hence, 10 seconds, the time required to pick up a nail, is worth $\frac{1}{9}$ of a cent.

Hence, since the value of the nail picked up is only $\frac{1}{40}$ of a cent, it does not pay the carpenter to pick up the nail, he losing thereby $\frac{1}{9}-\frac{1}{40}$ or $\frac{31}{360}$ of a cent.

It would not pay to pick up twenty-penny nails at the same rate.

115. Proposed by ALOIS F. KOVARIK, Instructor in Mathematics and Physics in Decorah Institute, Decorah, Ia.

Where shall a pole 120 feet high be broken so that the top may rest on the ground 40 feet from the foot? (Solve by arithmetic.)